Physics 4B
Chapter 23: Gauss' Law

"The only thing in life that is achieved without effort is failure." – Source unknown

"We are what we repeatedly do. Excellence, therefore, is not an act, but a habit." – Aristotle

"Act as if what you do makes a difference, because it does." – Source unknown

Reading: pages 605 – 620

Outline:

⇒ electric flux
⇒ Gauss' Law
⇒ Gauss' Law and Coulomb's Law
⇒ conductors in electrostatic equilibrium
⇒ applications of Gauss' Law
    cylindrical symmetry
    planar symmetry
    spherical symmetry

Problem Solving Techniques

You should know how to calculate the electric flux through a given surface. A useful fact to remember is: if the field is uniform, then the total flux through a closed surface is zero. Sometimes you can easily calculate the flux of a uniform field through part of a closed surface but you have been asked for the flux through the remaining part. Since the two contributions to the total flux sum to zero, they are the negatives of each other.

Some problems deal with Gauss' law in the form $\varepsilon_0\Phi = q_{enc}$, where Φ is the electric flux through a closed surface and q_{enc} is the charge enclosed by the surface. You might be given the charge and asked for the flux or you might be given the flux (or the information to calculate it) and asked for the enclosed charge.

When a problem deals with a conductor, you should remember that the electric field inside the conductor is 0, that the field just outside the conductor is perpendicular to the surface and has magnitude σ/ε_0, where σ is the area charge density, and that the total charge enclosed by any Gaussian surface completely inside the conductor is 0.

Many problems ask you to use Gauss' law to solve for the electric field in given situations. Become adept in finding an expression for the total flux through a Gaussian surface for the three special cases discussed in the text: cylindrical, spherical, and planar symmetry. For a cylindrical Gaussian surface, with a radial field that is uniform over the curved portion of the surface, $\int E \bullet dA = (2\pi rL)E$, where r is the radius of the Gaussian cylinder and L is its length. For a spherical Gaussian surface with a radial field that is uniform over the surface, $\int E \bullet dA = (4\pi r^2)E$, where r is the radius of the Gaussian sphere. Also remember that the electric field due to a large plane with uniform area charge density σ is $\sigma/2\varepsilon_0$.

Mathematical Skills

Electric Flux Calculations:

To calculate the electric flux, you should be familiar with the evaluation of simple area integrals. If the region of integration is a rectangle with sides *a* and *b*, you will probably want to position the coordinate system so that the rectangle is in the *x*, *y* plane with two of its sides along the axes. The infinitesimal element of area can be taken to be a rectangle with sides d*x* and d*y*. The flux integral then becomes

$$\Phi = \int_{x=0}^{a}\int_{y=0}^{b} E_z(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Integrations in *x* and *y* are carried out independently. If, for example, the *z* component of the electric field is given by $E_z = 9x^2y + 2y$, then

$$\Phi = \int_{x=0}^{a}\int_{y=0}^{b}\left(9x^2y^2 + 2y\right)\,\mathrm{d}x\,\mathrm{d}y = \int_{y=0}^{b}\left(3x^3y^2 + 2xy\right)\Big|_{x=0}^{a}\mathrm{d}y$$

$$= \int_{y=0}^{b}\left(3a^3y^2 + 2ay\right)\mathrm{d}y = \left(a^3y^3 + ay^2\right)\Big|_{y=0}^{b} = a^3b^3 + ab^2\ ,$$

where the integration over *x* was carried out first, followed by the integration over *y*.

For many problems of this chapter a Gaussian surface can be chosen so that the normal component of the electric field has the same value at all points on a portion of it and is zero on the other portions. The integral for the flux then reduces to $\Phi = EA$, where *A* is the area of the region over which the normal component of the field is not zero and *E* is the magnitude of the normal component.

To evaluate the flux, you must know how to calculate the areas of various surfaces. The area of a rectangle with sides of length *a* and *b* is *ab*; the area of a circle with radius *R* is πR^2; the surface area of a sphere with radius *R* is $4\pi R^2$; and the surface area of the rounded portion of a cylinder with radius *R* and length *L* is $2\pi RL$.

If you have trouble remembering that $2\pi rL$ gives the area of a cylindrical surface, imagine a paper towel roll wrapped exactly once around with a towel. The surface area of the roll is the same as the area of the towel. Since the towel is a rectangle with sides of length $2\pi r$ and *L*, its area is $2\pi rL$.

Calculations of charge:

You must also be able to calculate the charge enclosed by a Gaussian surface when the volume, area, or linear charge density is given. If an object has a uniform volume charge density ρ, for example, the charge enclosed is given by ρV, where V is the volume of the part of the object that lies within the Gaussian surface. If the Gaussian surface is completely within the object and the object does not have any cavities, then the region enclosed by the Gaussian surface is completely filled with charge and V is the volume enclosed by the Gaussian surface. If the object has a cavity that is wholly within the Gaussian surface, then V is the volume enclosed by the Gaussian surface minus the volume of the cavity.

Suppose, for example, a sphere of radius R has a uniform volume charge density ρ and the Gaussian surface is a concentric sphere of radius r. If $r < R$, then the Gaussian sphere is filled with charge and the charge enclosed is $4\pi\rho r^3/3$. If, on the other hand, $R < r$, then the Gaussian surface is only partially filled with charge. The charge enclosed is the total charge, or $4\pi\rho R^3/3$.

If a spherical object has a spherical cavity with radius R_c and the Gaussian surface is within the object but outside the cavity, then the charge enclosed is $\rho[(4\pi r^3/3) - (4\pi R_c^3/3)]$. The first term in the brackets is the volume enclosed by the Gaussian surface and the second is the volume of the cavity. If $R < r$, the charge enclosed is $\rho[(4\pi R^3/3) - (4\pi R_c^3/3)]$.

Now, consider a solid cylinder with radius R and length L, having a uniform volume charge density ρ. Suppose the Gaussian surface is a concentric cylinder with radius r and the same length as the cylinder of charge. If $r < R$, the charge enclosed is $2\pi\rho rL$ and if $R < r$, it is $2\pi\rho RL$. If the cylinder has a cavity that is inside the Gaussian surface, its volume must be subtracted from the volumes in these expressions.

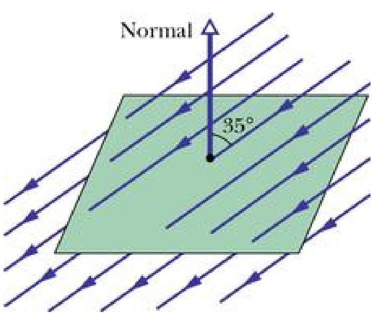
If the volume charge density varies from point to point in the object, you must evaluate the integral $\int \rho\, dV$ over the volume of that part of the object lying within the Gaussian surface. The most common example is a sphere with a charge density that varies only with distance r from the center. Carry out the integration by dividing the sphere into spherical shells with infinitesimal thickness dr. A typical shell extends from r to $r + dr$ and has a volume of $4\pi r^2\, dr$. Notice that this is the product of the surface area of the shell and its thickness. The charge in the shell is $4\pi\rho(r)r^2\, dr$. If the Gaussian surface is a sphere of radius r, concentric with the sphere of charge and entirely within it, the charge enclosed is $\int_0^r 4\pi\, \rho(r)\, r^2 dr$. If the Gaussian surface is entirely outside the charge distribution, the charge enclosed is $\int_0^R 4\pi\, \rho(r)\, r^2 dr$. Notice the upper limits of integration are different for these two cases.

For a cylinder with a charge density that depends only on the distance r from the axis, divide the cylinder into concentric cylindrical shells with thickness dr. The volume of a shell is $2\pi rL\, dr$, where L is the length of the cylinder. If the Gaussian surface is a concentric cylinder with radius r and is inside the charge distribution, then the charge enclosed is $\int_0^r 2\pi rL\, \rho(r)\, dr$. If the Gaussian surface is outside the distribution, the charge enclosed is $\int_0^R 2\pi rL\, \rho(r)\, dr$.

Questions and Example Problems from Chapter 23

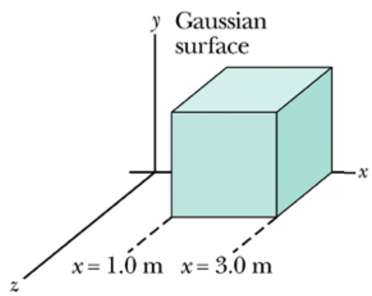
Question 1
A small charged ball lies within the hollow of a metallic spherical ball of radius R. For three situations, the net charges on the ball and shell, respectively, are (1) +4q, 0; (2) –6q, +10 q; (3) +16 q, -12q. Rank the situations according to the charge on (a) the inner surface of the shell and (b) the outer surface, most positive first.

Problem 1
The square surface shown in the figure below measures 3.2 mm on each side. It is immersed in a uniform electric field with magnitude E = 1800 N/C. The field lines make an angle of 35° with a normal to the surface, as shown. Take that normal to be directed "outward," as though the surface were one face of a box. Calculate the electric flux through the surface.



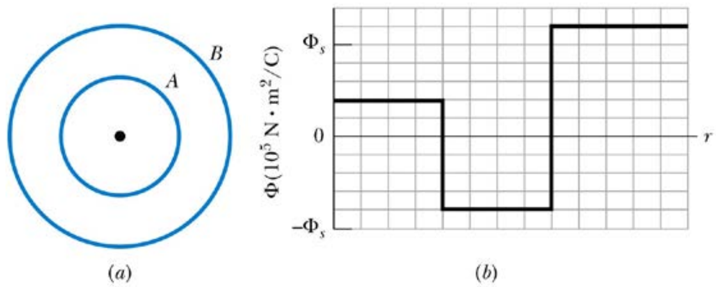
Problem 2

An electric field given by $E = 4.0\hat{i} - 3.0(y^2 + 2.0)\hat{j}$ pierces a Gaussian cube of length 2.0 m and positioned as shown in the figure below. (The magnitude of E is in N/C and position x is in m.) What is the electric flux through the (a) top face, (b) bottom face, (c) left face, and (d) back face? (e) What is the net electric flux through the cube?



Problem 3

A charged particle is suspended at the center of two concentric spherical shells that are very thin and made of nonconducting material. Figure a below shows a cross section. Figure b gives the net flux Φ through a Gaussian sphere centered on the particle, as a function of radius r of the sphere. The scale of the vertical axis is set by $\Phi s = 5.0 \times 10^5$ Nm^2/C. (A) What is the charge of the central particle? What are the net charges of (b) shell A and (c) shell B?

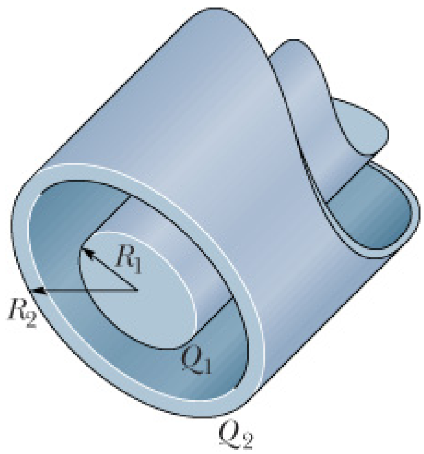


Problem 4

An isolated conductor of arbitrary shape has a net charge of $+10 \times 10^{-6}$ C. Inside the conductor is a cavity within which is a point charge $q = +3.0 \times 10^{-6}$ C. What is the charge (a) on the cavity wall and (b) on the outer surface of the conductor?

Problem 5

The figure below is a section of a conducting rod of radius R_1 = 1.30 mm and length L = 11.00 m inside a thin-walled coaxial conducting cylindrical shell of radius $R_2 = 10.0R_1$ and the (same) length L. The net charge on the rod is $Q_1 = +3.40 \times 10^{-12}$ C; that on the shell is $Q_2 = -2.00Q_1$. What are the (a) magnitude E and (b) direction (radially inward or outward) of the electric field at radial distance $r = 2.00R_2$? What are (c) E and (d) the direction at $r = 5.00R_1$? What is the charge on the (e) interior and (f) exterior surface of the shell?

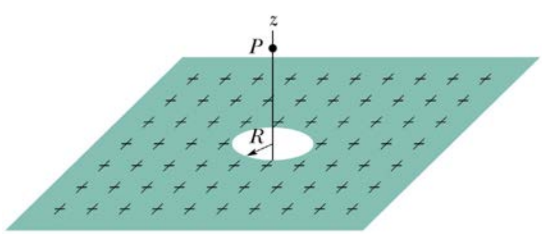


Problem 6

A square metal plate of edge length 8.0 cm and negligible thickness has a total charge of 6.0×10^{-6} C. (a) Estimate the magnitude E of the electric field just off the center of the plate (at, say, a distance of 0.50 mm) by assuming that the charge is spread uniformly over the two faces of the plate.

Problem 7

In the figure below, a small circular hole of radius R = 1.80 cm has been cut into the middle of an infinite flat, nonconducting surface that has uniform charge density $\sigma = 4.50$ pC/m^2. A z axis, with its origin at the hole's center, is perpendicular to the surface. In unit vector notation, what is the electric field at point P at z = 2.56 cm?



Problem 8

Two charged concentric spheres have radii of 10.0 cm and 15.0 cm. The charge on the inner sphere is 4.00×10^{-8} C, and that on the outer sphere is 2.00×10^{-8} C. Find the electric field (a) at r = 12.0 cm and (b) at r = 20.0 cm.

Problem 9

A solid nonconducting sphere of radius R = 5.60 cm has a nonuniform charge distribution of volume charge density $\rho = (14.1\ \text{pC/m}^3)\, r/R$, where r is the radial distance from the sphere's center. (a) What is the sphere's total charge? What is the magnitude E of the electric field at (b) r = 0, (c) r = R/2.00, and (d) r = R?